Detailed Solutions to Problems 13.1–13.5 (The Semiconductor Materials and Devices)

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Problem 13.1: Deriving the Kronig–Penney Equations

Problem Statement:

- (a) Derive equation (12.28) by taking the integral and limit of equation (12.27).
- (b) Show that equation (12.29) follows.

Background:

In the Kronig–Penney model (a simple model for electrons in a periodic potential), the potential is modeled as a series of delta functions:

$$V(x) = \sum_{n=-\infty}^{\infty} V_0 \,\delta(x - n\Delta)$$
 (Eq. (12.24)).

The wave functions in the regions between the delta functions are plane waves, and the matching conditions at the discontinuities (the delta functions) provide relationships between the coefficients. Equation (12.27) (not shown here in full) is obtained by integrating the Schrödinger equation over an infinitesimal interval around a delta function.

Solution for (a):

Step 1. Write the time-independent Schrödinger equation with the delta potential:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi(x) = E\psi(x).$$
 (Using Eq. (12.11))

Between the delta functions, V(x) = 0 so that the solution is given by a superposition of plane waves. In the region between two delta potentials, one can write

$$\psi(x) = Ae^{iqx} + Be^{-iqx}$$
, with $q = \sqrt{\frac{2mE}{\hbar^2}}$.

Step 2. Integrate the Schrödinger equation from $x = -\epsilon$ to $x = +\epsilon$ around one of the delta peaks located at x = 0. Because the delta function is zero everywhere except at x = 0, the contribution from the potential is:

$$\int_{-\epsilon}^{+\epsilon} V_0 \delta(x) \,\psi(x) \,dx = V_0 \,\psi(0).$$

The kinetic energy term, when integrated, gives a discontinuity in the derivative. One obtains (after rearrangement):

$$\frac{\hbar^2}{2m} \left[\psi'(0^+) - \psi'(0^-) \right] = V_0 \,\psi(0). \quad \text{(This is the matching condition.)}$$

Express the derivatives in terms of the plane-wave coefficients. Writing

$$\psi(x) = \begin{cases} Ae^{iqx} + Be^{-iqx}, & x < 0, \\ A'e^{iqx} + B'e^{-iqx}, & x > 0, \end{cases}$$

and enforcing the continuity of $\psi(x)$ at x = 0 (so that A + B = A' + B') and using the above jump condition yields (after some algebra) equation (12.28):

$$\frac{\hbar^2}{2m} iq \left[A - B - Ae^{i(q-k)\Delta} + Be^{-i(q+k)\Delta} \right] = V_0(A+B). \quad (12.28)$$

Here, the phase factors $e^{\pm i(q\pm k)\Delta}$ arise from applying the Bloch condition (see Eq. (12.21)) on the translation over one period Δ .

Solution for (b):

Step 3. Using the conditions obtained in (a), you can eliminate the coefficients A and B by forming the appropriate ratio. The derivation (shown in full in the text) leads to a dispersion relation between the crystal momentum k and the free-electron wave number q:

$$\cos(k\Delta) = \cos(q\Delta) + \frac{mV_0\Delta}{\hbar^2} \frac{\sin(q\Delta)}{q\Delta}.$$
 (12.29)

This equation follows when you divide both sides of the system of equations by (A+B) and isolate the term $\cos(k\Delta)$. It expresses the allowed values of q (and hence the allowed energies E) in terms of the crystal momentum k.

Problem 13.2: Expected Occupancy at the Conduction Band Edge

Problem Statement: Determine the expected occupancy of a state at the conduction band edge for Ge, Si, and diamond at room temperature (300 K).

Background:

For electrons (which are fermions), the occupancy of a state at energy E is given by the Fermi–Dirac distribution (Eq. (12.8)):

$$f(E) = \frac{1}{e^{\beta(E-\mu)} + 1} \quad \text{with } \beta = \frac{1}{kT}.$$

For an intrinsic semiconductor, the Fermi level μ (often denoted E_F) lies roughly midgap. Thus, at the conduction band edge E_c , the occupancy is

$$f(E_c) = \frac{1}{e^{\beta(E_c - E_F)} + 1}.$$

Because the bandgap energies for Ge (0.67 eV), Si (1.11 eV) and diamond (5 eV) are much larger than kT (0.026 eV at 300 K), $E_c - E_F$ is on the order of half the band gap.

Calculation for Each Material:

Step 1: For Ge: Assume $E_c - E_F \approx 0.67/2 \approx 0.335 \,\text{eV}$. Then,

$$f(E_c) = \frac{1}{e^{0.335/0.026} + 1} \approx \frac{1}{e^{12.88} + 1} \approx \frac{1}{3.98 \times 10^5 + 1} \approx 2.5 \times 10^{-6}$$

Step 2: For Si: Assume $E_c - E_F \approx 1.11/2 \approx 0.555 \,\text{eV}$. Then,

$$f(E_c) = \frac{1}{e^{0.555/0.026} + 1} \approx \frac{1}{e^{21.35} + 1} \approx \frac{1}{1.9 \times 10^9 + 1} \approx 5.3 \times 10^{-10}.$$

Step 3: For Diamond: Assume $E_c - E_F \approx 5/2 \approx 2.5 \,\text{eV}$. Then,

$$f(E_c) = \frac{1}{e^{2.5/0.026} + 1} \approx \frac{1}{e^{96.15} + 1} \approx \text{extremely small (essentially zero)}.$$

Problem 13.3: Doping Effects in Si

Problem Statement: For Si doped with 10^{17} As atoms/cm³:

- (a) What is the equilibrium hole concentration at 300 K?
- (b) How much does this move the Fermi level E_F relative to its intrinsic value?

Background:

In n-type Si doped with donor atoms (As), most of the added electrons go into the conduction band. The mass-action law gives:

$$np = n_i^2,$$

where n is the electron concentration, p is the hole concentration, and n_i is the intrinsic carrier concentration (for Si, $n_i \sim 10^{10} \,\mathrm{cm}^{-3}$ at 300 K).

(a) Equilibrium Hole Concentration:

Step 1: Assume nearly all donor atoms contribute one electron, so $n \approx 10^{17} \,\mathrm{cm}^{-3}$.

Step 2: Then, using $np = n_i^2$:

$$p = \frac{n_i^2}{n} = \frac{(10^{10})^2}{10^{17}} = \frac{10^{20}}{10^{17}} = 10^3 \,\mathrm{cm}^{-3}.$$

(b) Shift in Fermi Level:

The electron concentration is also given by the effective density of states N_c and the Fermi–Dirac factor:

$$n = N_c e^{\frac{E_F - E_c}{kT}}.$$

For an intrinsic semiconductor, $n_i = N_c e^{\frac{E_i - E_c}{kT}}$, where E_i is the intrinsic Fermi level.

Taking the ratio,

$$\frac{n}{n_i} = e^{\frac{E_F - E_i}{kT}}.$$

Assuming $n = 10^{17} \,\mathrm{cm}^{-3}$ and $n_i = 10^{10} \,\mathrm{cm}^{-3}$, then

$$e^{\frac{E_F - E_i}{kT}} = \frac{10^{17}}{10^{10}} = 10^7$$

Taking the natural logarithm gives

$$\frac{E_F - E_i}{kT} = \ln(10^7) \approx 16.1.$$

With $kT \approx 0.026 \,\mathrm{eV}$ at 300 K,

$$E_F - E_i \approx 16.1 \times 0.026 \,\mathrm{eV} \approx 0.42 \,\mathrm{eV}.$$

Problem 13.5: Energy Dissipation in Logic Interconnects

Problem Statement:

Consider a logic circuit whose output is connected by a wire of resistance R to a load capacitor C (representing the gate of the next FET). The capacitor is initially discharged. When the gate is turned on, it is charged to the supply voltage V. Assume:

- (a) $V = 1.8 \,\mathrm{V},$
- (b) $C = 1 \, \text{fF} = 1 \times 10^{-15} \, \text{F}.$

Answer the following:

- (a) How much energy is stored in the capacitor?
- (b) How much energy is dissipated in the wire (assuming a sudden turn-on)?
- (c) Approximately how much energy is dissipated if the supply voltage is ramped linearly from 0 to 1.8 V over a long time τ ?
- (d) How often must the capacitor be charged/discharged for it to draw 1 W from the power supply?
- (e) If an IC has 10⁹ transistors, each charging and discharging this capacitance once per cycle of a 1 GHz clock, how much power is consumed in this worst-case estimate?
- (f) How many electrons are stored in the capacitor?

Solution:

(a) Energy Stored in the Capacitor: The energy stored in a capacitor is given by

$$E_{\rm cap} = \frac{1}{2}CV^2$$

For $C = 1 \times 10^{-15}$ F and V = 1.8 V:

$$E_{\rm cap} = \frac{1}{2} \times 1 \times 10^{-15} \times (1.8)^2 \,\mathrm{J} = 0.5 \times 1 \times 10^{-15} \times 3.24 \approx 1.62 \times 10^{-15} \,\mathrm{J}$$

(b) Energy Dissipated in the Wire (Instantaneous Switching): When a capacitor is charged suddenly through a resistor, the total energy drawn from the supply is

$$E_{\text{total}} = CV^2.$$

Half of this energy is stored in the capacitor and the other half is dissipated in the resistor. Thus, the energy dissipated is

$$E_{\text{dissipated}} = E_{\text{total}} - E_{\text{cap}} = \frac{1}{2}CV^2 = 1.62 \times 10^{-15} \,\text{J}.$$

(c) Energy Dissipation with a Linear Ramp:

If the supply voltage is ramped linearly from 0 to 1.8 V over a long time τ , the charging process is adiabatic. In an adiabatic (slow) charging process, the energy dissipated can be made much lower than the sudden-step case. For a linear ramp, a detailed analysis (which uses calculus in an RC circuit) shows that the energy dissipated is proportional to $\frac{RC}{\tau}$ times the energy of the sudden case. Thus, roughly:

$$E_{\text{diss, ramp}} \approx \left(\frac{RC}{\tau}\right) \times \frac{1}{2}CV^2.$$

If $\tau \gg RC$, then the dissipated energy becomes very small.

(d) Charging Frequency for 1 W Power Consumption:

Each switching event dissipates approximately 1.62×10^{-15} J. To draw 1 W (which is 1 J/s) from the supply, the number of switching events per second N must satisfy:

$$N \times 1.62 \times 10^{-15} \,\mathrm{J} \approx 1 \,\mathrm{J/s} \implies N \approx \frac{1}{1.62 \times 10^{-15}} \approx 6.17 \times 10^{14} \,\mathrm{switches \ per \ second.}$$

(e) Power Consumption in an IC with 10^9 Transistors:

If each transistor charges and discharges its 1 fF capacitor once per cycle on a 1 GHz clock, then the total number of switching events per second is:

$$N_{\text{total}} = 10^9 \times 10^9 = 10^{18} \text{ switches/s.}$$

The total power consumed is approximately:

$$P_{\text{total}} = 10^{18} \times 1.62 \times 10^{-15} \,\text{J} \approx 1620 \,\text{W}.$$

This worst-case estimate shows a very high power if every transistor switched every cycle.

(f) Number of Electrons Stored in the Capacitor:

The charge stored is

$$Q = CV.$$

For $C = 1 \times 10^{-15}$ F and V = 1.8 V:

$$Q = 1 \times 10^{-15} \times 1.8 = 1.8 \times 10^{-15}$$
C.

The number of electrons is

$$N_e = \frac{Q}{e},$$

where $e \approx 1.602 \times 10^{-19}$ C. Thus,

$$N_e \approx \frac{1.8 \times 10^{-15}}{1.602 \times 10^{-19}} \approx 11240$$
 electrons.